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J. Math. Anal. Appl. 303 (2005) 342–349
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On a nonlinear eigenvalue problem in ODE

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Received 8 June 2002
Available online 18 September 2004
Submitted by P.G.L. Leach

Abstract

In this paper we shall study the following variant of the logistic equation with diffusion:

$$-du''(x) = g(x)u(x) - u^2(x)$$

for $x \in R$. The unknown function u corresponds to the size of a population. The function g corresponds to the birth (or death) rate of the population which takes on both positive and negative values on R; the $-u^2$ term in the equation corresponds to the fact that the population is self-limiting and the parameter $d > 0$ corresponds to the rate at which the population diffuses. We have obtained our results by the construction of sub and supersolutions and the study of asymptotic properties of solutions. Our results show the interplay between the birth rate of the species and the extent of diffusion in determining the existence or nonexistence of nontrivial steady-state distributions of population.

Keywords: Sub and supersolutions; Principal eigenvalue; Variational characterisations

1. Introduction

We shall study the following variant of the logistic equation with diffusion:

$$u_t(x,t) = d\Delta u(x,t) + g(x)u(x) - u^2(x) \tag{1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.08.040

for $x \in R$, $t > 0$. The unknown function u corresponds to the density of a population. The parameter $d > 0$ corresponds to the rate at which the population diffuses. If we write $\lambda = 1/d$, we see that steady-state solutions of (1) must satisfy

$$-u''(x) = \lambda\big[g(x)u(x) - u^2(x)\big] \tag{2}$$

for $x \in R$. The $-u^2$ term in the equation corresponds to the fact that the population is self-limiting and the function g corresponds to the birth rate of the population if self-limitation is ignored. We assume throughout that g is a smooth function which takes on both positive and negative values on R; at points where $g(x) > 0$ (< 0) the population, ignoring self-limitation, has positive (negative) birth rate. Because u represents a population density, we investigate only nonnegative solutions. Such solutions correspond to possible steady-state distributions of population.

In Section 2 by constructing sub and supersolutions we prove an existence result and also prove under the strong restriction that $g(x) < 0$ in a neighbourhood of infinity that every positive solution must tend to 0 as $|x| \to \infty$. In Section 3 under the weaker assumption that g^+ is small at infinity ($g^+(x) \leqslant k/|x|^{2+\delta}$) we again prove that every positive solution must tend to 0 as $|x| \to \infty$ and also establish nonexistence and uniqueness results.

Diffusive logistic equations on all of R^N were previously studied in the periodic-parabolic case by Koch Medina and Schatti [7] and Hieber et al. [6] under the assumption that g is negative at infinity in the dynamical systems framework of spaces of bounded uniformly continuous functions.

Steady-state solutions of (1) on a bounded region with Neumann boundary conditions were recently studied by Cantrell et al. [5], and on all of R^N for $N \geqslant 3$ by Afrouzi and Brown [2].

2. Existence of solutions

First we assume that $g : R \to R$ is a smooth function which is bounded above and takes on both positive and negative values in R and we prove the existence of positive solutions by constructing appropriate sub-and supersolutions. The existence of a solution lying between a sub-and supersolution is a standard result for a bounded domain. It is, however, easy to obtain the same result when the bounded region is replaced by all of R by considering a sequence of intervals of increasing radius and using a subsequence argument (see, e.g., [4]).

It is easy to see that, since g is bounded above, then sufficiently large positive constants are supersolutions of (2).

We now show how to construct subsolutions. Let $r > 0$, $B_r = (-r, r)$ and consider the eigenvalue problem

$$\begin{cases} -u''(x) = \lambda g(x)u(x), & x \in (-r, r), \\ u(-r) = 0 = u(r). \end{cases} \tag{3}$$

If g takes on positive values for at least some points in B_r, then (3) has a positive principal eigenvalue which we denote by $\lambda_1^+(r)$. It is also well known (see, e.g., [3]) that

$$\lambda_1^+(r) = \inf\left\{ \frac{\int_{-r}^r u'^2\,dx}{\int_{-r}^r gu^2\,dx} \colon\ u \in H_0^1((-r,r)),\ \int_{-r}^r gu^2\,dx > 0 \right\}.$$

It is easy to see that $r \to \lambda_1^+(r)$ is a decreasing function and so $\lim_{r\to+\infty}\lambda_1^+(r)$ must exist. Suppose $\lambda_\infty = \lim_{r\to\infty}\lambda_1^+(r)$. Then $\lambda_\infty \geqslant 0$ and λ_∞ has the following variational characterisations (see [1, Lemmas 2.7 and 2.8]):

$$\lambda_\infty = \inf\left\{ \frac{\int_R u'^2\,dx}{\int_R gu^2\,dx} \colon\ u \in C_0^\infty(R),\ \int_R gu^2\,dx > 0 \right\}$$

and

$$\lambda_\infty = \inf\left\{ \frac{\int_R u'^2\,dx}{\int_R gu^2\,dx} \colon\ u \in H^1(R),\ u \text{ has compact support},\ \int_R gu^2\,dx > 0 \right\}.$$

Now suppose that $\lambda > \lambda_\infty$. Then we may choose r s.t. $\lambda_1^+(r) < \lambda$. We will define $\underline{u}$ by

$$\underline{u}(x) = \begin{cases} \phi(x) & \text{if } x \in (-r,r), \\ 0 & \text{otherwise,} \end{cases}$$

where ϕ is the positive eigenfunction corresponding to $\lambda_1^+(r)$. It is easy to see that $\epsilon\underline{u}$ is a subsolution of (2) for arbitrarily small $\epsilon > 0$.

We now show that $\psi(x) = M/(1+x^2)$ is also a supersolution of (2). We have $\psi'(x) = -2Mx/(1+x^2)^2$ and so

$$\psi'' = \frac{-2M(1+x^2)^2 + 8Mx^2(1+x^2)}{(1+x^2)^4} = \frac{-2M(1+x^2) + 8Mx^2}{(1+x^2)^3}.$$

Then we have

$$\begin{aligned} -\psi''(x) &= \frac{2M}{(1+x^2)^2} - \frac{8Mx^2}{(1+x^2)^3} = \frac{M}{1+x^2}\left[\frac{2}{1+x^2}\left(1 - \frac{4x^2}{1+x^2}\right)\right] \\ &= \psi(x)\left[\frac{2}{1+x^2}\left(1 - \frac{4x^2}{1+x^2}\right)\right]. \end{aligned}$$

Hence ψ is a supersolution iff

$$\psi(x)\frac{2}{1+x^2}\left[1 - \frac{4x^2}{1+x^2}\right] \geqslant \lambda g(x)\psi(x) - \lambda\left[\psi(x)\right]^2,$$

i.e.,

$$\frac{2}{1+x^2}\left[1 - \frac{4x^2}{1+x^2}\right] \geqslant \lambda g(x) - \lambda\frac{M}{1+x^2},$$

i.e.,

$$\frac{2}{1+x^2}\left[1 + \frac{\lambda M}{2} - \frac{4x^2}{1+x^2}\right] \geqslant \lambda g(x) \tag{4}$$

for $x \in R$. Since $g(x) \leqslant k$ for some constant $k > 0$, it is easy to see that for any given $\lambda > 0$ it is possible to choose M sufficiently large so that (4) is satisfied. The following theorem is a trivial consequence of the above results

Theorem 1. *There exists a positive solution of* (2) *such that* $u \to 0$ *at* ∞ *whenever* $\lambda > \lambda_\infty$.

Theorem 2. *Suppose that there exists* $r_0 > 0$ *such that* $g(x) < 0$ *whenever* $|x| \geqslant r_0$. *If* u *is a positive solution of* (2), *then* $\lim_{|x|\to\infty} u(x) = 0$.

Proof. First we assume that u is bounded; since $-u''(x) = \lambda[g - u]u$, we have $u''(x) > 0$ for x such that $|x| \geqslant r_0$. Hence $u'(x)$ is an increasing function of x, for $|x| \geqslant r_0$ and so $u'(x)$ is eventually of one sign. Thus u is eventually a monotone function and so, since u is a bounded function, $\lim_{|x|\to\infty} u(x)$ exists. For x such that $|x| > r_0$, we have $-u'' \leqslant -\lambda u^2$ and so $u'' \geqslant \lambda u^2$. By integrating we obtain

$$u'(x) - u'(r_0) \geqslant \lambda \int_{r_0}^{x} u^2(x)\,dx. \tag{5}$$

Thus, if $\lim_{|x|\to\infty} u(x) \neq 0$, letting $|x| \to \infty$ in (5), we see that $\lim_{|x|\to\infty} u'(x) = \infty$. This is impossible as u is bounded and so we must have $\lim_{|x|\to\infty} u(x) = 0$.

Now we shall prove that every solution of (2) must $\to 0$ as $|x| \to \infty$. It is sufficient to prove that every positive solution of (2) must be bounded for all x such that $|x| \geqslant r_0$. Hence this implies that for x, $|x| \geqslant r_0$, every turning point of u must be a local minimum and so u must be eventually monotone.

Now suppose that u is not bounded. Since u is eventually monotone, it follows that $\lim_{|x|\to\infty} u(x) = \infty$, and that, for x such that $|x|$ is sufficiently large, e.g., $|x| \geqslant r_1$, we have $u'(x) > 0$ and so $u''u' > \lambda u'u^2$. Therefore we have

$$\left(\frac{1}{2}(u')^2 - ku^3\right)' \geqslant 0$$

for some constant $k > 0$ whenever $|x| \geqslant r_1$. Hence $\frac{1}{2}(u')^2 \geqslant c + ku^3$, where c is a constant, and so, as $\lim_{|x|\to\infty} u(x) = \infty$, we have

$$u' > k_1 u^{3/2}$$

for some positive constant k_1 whenever $|x| \geqslant r_1$. Thus $u'u^{-3/2} > k_1$ and so

$$-2u^{-1/2}(-r_1) + 2u^{-1/2}(x) \geqslant k(-x - r_1) \tag{6}$$

for all $x \leqslant -r_1$ which is a contradiction since $\lim_{|x|\to\infty} u(x) = \infty$. Similarly

$$-2u^{-1/2}(x) + 2u^{-1/2}(r_1) \geqslant k(x - r_1) \tag{7}$$

for all $x \geqslant r_1$, which is also a contradiction since $\lim_{|x|\to\infty} u(x) = \infty$. Hence we have shown that every positive solution of (2) must be bounded and so $\lim_{|x|\to\infty} u(x) = 0$. □

3. Uniqueness and nonexistence of solutions

For the remainder of the paper we shall assume that there exists $k > 0$, $\delta > 0$ and $r_1 > 0$ such that

$$g^+(x) \leqslant \frac{k}{|x|^{2+\delta}}, \quad |x| \geqslant r_1.$$

Theorem 3. *Suppose u is a positive solution of* (2) *such that* $\lim_{|x|\to\infty} u(x) = 0$. *Then*

$$u, u' \in L^2(R) \quad \textit{and} \quad gu \in L^1(R).$$

Proof. For $x \in [r_1, +\infty)$ we have

$$\int_{r_1}^{x} g^+ u\,dx \leqslant k \int_{r_1}^{x} \frac{1}{t^{2+\delta}}\,dt = \frac{k}{-1-\delta}\left[t^{-1-\delta}\right]_{r_1}^{x} = \frac{-k}{1+\delta}\left[\frac{1}{x^{1+\delta}} - \frac{1}{r_1^{1+\delta}}\right]$$

and since

$$\lim_{x\to\infty} \frac{k}{1+\delta}\left[\frac{1}{r_1^{1+\delta}} - \frac{1}{x^{1+\delta}}\right] = \frac{k}{(1+\delta)r_1^{1+\delta}},$$

we have $g^+u \in L^1[0, +\infty)$.

Now suppose that $g^-u + u^2 \notin L^1[0, +\infty)$. We have

$$u'' = -\lambda(g^+ - g^- - u)u = -\lambda g^+ u + \lambda(g^- u + u^2)$$

and so

$$\int_{r_1}^{x} u'' = -\lambda \int_{r_1}^{x} g^+ u + \lambda \int_{r_1}^{x} (g^- u + u^2).$$

Since $g^-u + u^2 \notin L^1[0, +\infty)$, we obtain that $\lim_{|x|\to\infty} u'(x) = +\infty$ which is a contradiction. Thus we must have $g^-u \in L^1[0, +\infty)$ and $u \in L^2[0, +\infty)$. Now we show that $u' \in L^2[0, +\infty)$. We have

$$\int_{r_1}^{x} u'' = -\lambda \int_{r_1}^{x} (gu - u^2)$$

and so

$$u'(x) = u'(r_1) - \lambda \int_{r_1}^{x} (gu - u^2).$$

Thus $\lim_{x\to+\infty} u'(x)$ exists. Also as $u'' = -\lambda(gu - u^2)$ we have

$$uu'' = -\lambda(gu - u^2)u = -\lambda(g^+ u - g^- u - u^2)u = \lambda(g^+ u)u + \lambda(g^- u + u^2)u$$

and since $g^+u \in L^1[0,+\infty)$, $g^-u + u^2 \in L^1[0,+\infty)$, we have $uu'' \in L^1[0,+\infty)$. We also have

$$\int_{r_1}^{x} uu'' = -\lambda \int_{r_1}^{x} (gu - u^2)u$$

and so

$$u(x)u'(x) - u(r_1)u'(r_1) - \int_{r_1}^{x} (u')^2 = -\lambda \int_{r_1}^{x} (gu - u^2)u.$$

Since $\lim_{x\to+\infty} u(x) = 0$, $\lim_{x\to+\infty} u'(x)$ exists and $(gu - u^2)u \in L^1[0,+\infty)$, it follows that $u' \in L^2[0,+\infty)$. By a quite similar argument it can be shown that

$$u, u' \in L^2(-\infty, 0] \quad \text{and} \quad gu \in L^1(-\infty, 0].$$

In fact we have proved that

$$u, u' \in L^2(R) \quad \text{and} \quad gu \in L^1(R). \qquad \square$$

Theorem 4. *There exists $\lambda' > 0$ such that if $\lambda < \lambda'$, then there does not exist a positive solution of* (2) *such that* $\lim_{x\to+\infty} u(x) = 0$.

Proof. Suppose that u is a positive solution of (2) such that $\lim_{|x|\to+\infty} u(x) = 0$. We have

$$\int_0^x (u')^2 = u(x)u'(x) - u(0)u'(0) + \lambda \int_0^x (gu - u^2)u$$

and so there exists $k_1 > 0$ such that

$$\int_0^\infty (u')^2 < \lambda k_1 \int_0^\infty gu^2.$$

Similarly we can show there exists $k_2 > 0$ such that

$$\int_{-\infty}^0 (u')^2 < \lambda k_2 \int_{-\infty}^0 gu^2.$$

Now by choosing $k' = \max\{k_1, k_2\}$ we have

$$\int_{-\infty}^{+\infty} (u')^2 < \lambda k' \int_{-\infty}^{+\infty} gu^2. \tag{8}$$

Since $u, u' \in L^2(R)$, so $u \in W^{1,2}(R)$. It is known that

$$\int_R (\varphi')^2 \geqslant \lambda_\infty \int_R g\varphi^2\, dx \tag{9}$$

for all $\varphi \in W^{1,2}(R)$ with compact support such that $\int_R g\varphi^2\,dx > 0$. It is easy to obtain from (8) and (9) that $\lambda k' \geqslant \lambda_\infty$, i.e., $\lambda \geqslant \frac{1}{k'}\lambda_\infty = \lambda'$, and so the proof is complete. $\square$

Theorem 5. *There exists at most one positive solution of* (2) *such that* $\lim_{|x|\to\infty} u(x) = 0$.

Proof. Suppose that u and v are two distinct such solutions . If neither $u \leqslant v$ nor $v \leqslant u$, we define $s(x) = \inf\{u(x), v(x)\}$. Then $s \not\equiv u$ and $s \not\equiv v$ and s is a supersolution for (2). On the other hand, as before we can construct an arbitrarily small subsolution and so there must exists a solution w of (2) such that $w \leqslant s$. Clearly $w \leqslant u$ and $w \leqslant v$. Thus we may assume without loss of generality that $u \geqslant v$ and $u \not\equiv v$. Multiplying the u-equation by v, the v equation by u, integrating over $(-x, x)$ and subtracting gives

$$v(x)u'(x) - v(-x)u'(-x) - u(x)v'(x) + u(-x)v'(-x) = \lambda \int_{-x}^{x} uv(u - v).$$

Letting $|x| \to +\infty$ gives $\int_{-\infty}^{+\infty} uv(u - v) = 0$ and this is a contradiction. $\square$

Theorem 6. *If u is a bounded positive solution of* (2), *then*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Proof. It is easy to see that either $u \to 0$ or u is eventually monotone. We assume $u \not\to 0$. As $-u'' = \lambda g u - \lambda u^2 \leqslant \lambda g^+ u - \lambda u^2$ we have

$$u'(x) - u'(r_1) \geqslant -\lambda \int_{r_1}^{x} g^+ u + \lambda \int_{r_1}^{x} u^2.$$

Since $\int_{r_1}^{x} g^+ u$ is converges, we obtain $\lim_{|x|\to\infty} u'(x) = +\infty$, and this a contradiction as u is bounded. $\square$

Theorem 7. *Every positive solution of* (2) *is bounded.*

Proof. Suppose u is unbounded. Then u is eventually monotone increasing, i.e., $u(x) \to +\infty$ as $x \to +\infty$ or $x \to -\infty$. We prove the theorem for the case that $u(x) \to +\infty$ as $x \to +\infty$, the case $u(x) \to +\infty$ as $x \to -\infty$ is similar. Let $v(x) = e^{x/2}u(x)$. Since u is eventually monotone increasing, so is v. Thus there exists $r' \in R$ such that $v''(x) > v(x)$ for all $x > r'$. Then choosing $k > 0$ such that $v(x) - ke^x > 0$ and $[v(x) - ke^x]' > 0$ when $x = r'$, it is easy to show that $v(x) > ke^x$ for all $x \geqslant r'$. Hence, if $x \geqslant r'$,

$$v'' > ce^{-x/2}v^2 > ce^{-x/2}k^{1/2}e^{x/2}v^{3/2} = k'v^{3/2}$$

for some constant $k' > 0$. Thus, as $v' > 0$, we must have

$$v''v' > k'v^{3/2}v',$$

and so

$$\left[\frac{1}{2}(v')^2 - \frac{2k'}{5}v^{5/2}\right]' \geqslant 0.$$

Hence

$$(v')^2 \geqslant a + bv^{5/2}$$

for $x \geqslant r'$, where a and $b > 0$ are constants. Because v is unbounded, it follows that there exists $r'' \geqslant r'$ and $c > 0$ such that

$$v' \geqslant cv^{5/4},$$

where $x \geqslant r''$. Thus $v^{-5/4}v' > c$ and so $4(v^{-1/4}(c) - v^{-1/4}(x)) \geqslant c(x - c)$ for all $x \geqslant c$. Since $\lim_{x\to\infty} v(x) = \infty$, this is a contradiction and so we may conclude that u is bounded. □

Now from Theorems 6 and 7 we have

Theorem 8. *If u is a positive solution of* (2), *then*

$$\lim_{|x|\to\infty} u(x) = 0.$$

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